Public Key Cryptography

Applications Algorithms and Mathematical Explanations

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Abstract: The paper discusses public key cryptography and its use in applications such as Key Agreement, Data Encryption and Digital Signature. The paper discusses some public key algorithms such as DH, RSA, DSA, ECDH and ECDSA and also gives mathematical explanations on the working of these algorithms. The paper also gives a brief introduction to modular arithmetic, which is the core arithmetic of almost all public key algorithms.

1. Introduction

The data transferred from one system to another over public network can be protected by the method of encryption. On encryption the data is encrypted/scrambled by any encryption algorithm using the 'key'. Only the user having the access to the same 'key' can decrypt/de-scramble the encrypted data. This method is known as private key or symmetric key cryptography. There are several standard symmetric key algorithms defined. Examples are AES, 3DES etc. These standard symmetric algorithms defined are proven to be highly secured and time tested. But the problem with these algorithms is the key exchange. The communicating parties require a shared secret, 'key', to be exchanged between them to have a secured communication. The security of the symmetric key algorithm depends on the secrecy of the key. Keys are typically hundreds of bits in length, depending on the algorithm used. Since there may be number of intermediate points between the communicating parties through which the data passes, these keys cannot exchanged online in a secured manner. In a large network, where there are hundreds of system connected, offline key exchange seems too difficult and even unrealistic. This is where public key cryptography comes to help. Using public key algorithm a shared secret can be established online between communicating parties with out the need for exchanging any secret data.

In public key cryptography each user or the device taking part in the communication have a pair of keys, a public key and a private key, and a set of operations associated with the keys to do the cryptographic operations. Only the particular user/device knows the private key whereas the public key is distributed to all users/devices taking part in the communication. Since the knowledge of public key does not compromise the security of the algorithms, it can be easily exchanged online.

A shared secret can be established between two communicating parties online by exchanging only public keys and public constants if any. Any third party, who has access only to the exchanged public information, will not be able to calculate the shared secret unless it has access to the private key of any of the communicating parties. This is key agreement and is defined in section 2.

Apart from Key Agreement the other important applications of public key cryptography are Data Encryption and Digital Signature, which are explained in sections 3 and 4 respectively.

1.1. One-Way function

In public key cryptography, keys and messages are expressed numerically and the operations are expressed mathematically. The private and public key of a device is related

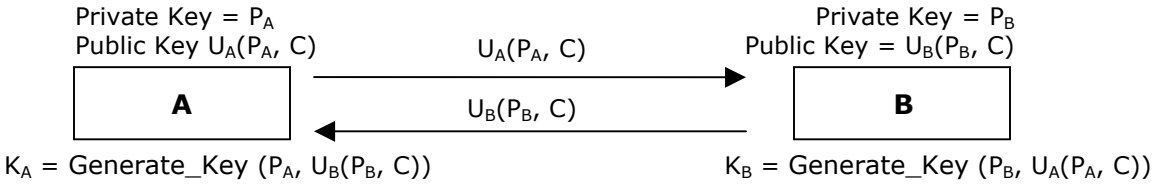
by the mathematical function called the one-way function. One-way functions are mathematical functions in which the forward operation can be done easily but the reverse operation is so difficult that it is practically impossible. In public key cryptography the public key is calculated using private key on the forward operation of the one-way function. Obtaining of private key from the public key is a reverse operation. If the reverse operation can be done easily, that is if the private key is obtained from the public key and other public data, then the public key algorithm for the particular key is cracked. The reverse operation gets difficult as the key size increases. The public key algorithms operate on sufficiently large numbers to make the reverse operation practically impossible and thus make the system secure. For e.g. RSA algorithm operates on large numbers of thousands of bits long.

2. Key Agreement

Key agreement is a method in which the device communicating in the network establishes a shared secret between them without exchanging any secret data. In this method the devices that need to establish shared secret between them exchange their public keys. Both the devices on receiving the other device's public key performs key generation operation using its private key to obtain the shared secret.

As we see in the previous section the public keys are generated using private key and other shared constants. Let P be the private key of a device and U(P, C) be the public key. Since public key is generated using private key, the representation U(P, C) shows that the public key contain the components of private key P and some constants C where C is known by all the device taking part in the communication.

Consider two devices A and B. Let P_A and $U_A(P_A, C)$ be the private key and public key of device A, and P_B and $U_B(P_B, C)$ be the private key and public key of device B respectively. Both device exchanges their public keys.
Device A, having got the public key of B, uses its private key to calculate shared secret K_A=Generate_Key(P_A, $U_B(P_B, C)$)
Device B, having got the public key of A, uses its private key to calculate the shared secret K_B=Generate_Key(P_B, $U_A(P_A, C)$)

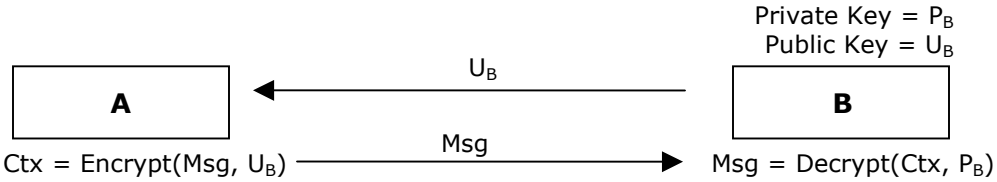


The key generation algorithm 'Generate_Key' will be such that the generated keys at the device A and B will be the same, that is shared secret K_A=K_B=K(P_A, P_B, C).
Since it is practically impossible to obtain private key from the public key any middleman, having access only to the public keys $U_A(P_A, C)$ and $U_B(P_B, C)$, will never be able to obtain the shared secret K.
Examples of key agreement algorithms are DH, RSA and ECDH. The algorithms and explanations are given in sections 6, 7.2 and 10 respectively.
During the key exchange process the public keys may pass through different intermediate points. Any middleman can thus tamper or change the public keys to its public key. Therefore for establishing shared secret it is important that device A receives the correct public key from device B and vice versa. Digital Certificate helps to deliver the public key in authenticated method. Digital Certificate is explained in section 4.1.

3. Encryption

Encryption is a process in which the sender encrypts/scrambles the message in such a way that only the recipient will be able to decrypt/ descramble the message.

Consider a device B whose private key and public key are P_B and U_B respectively. Since U_B is public key all devices will be able to get it. For any device that needs to send the message 'Msg' in a secured way to device B, it will encrypt the data using B's public key to obtain the cipher text 'Ctx'. The encrypted message, cipher text, can only be decrypted using B's private key. On receiving the message the B decrypts it using its private key P_B. Since only B knows its private key P_B none other including A can decrypt the message.



It is important that device A receives the correct public key from device B, i.e. no middleman must tamper or change the public key to its public key. Digital Certificate helps to deliver the public key in authenticated method. Digital Certificate is explained in section 4.1.

One of the popular public key encryption algorithms is RSA. RSA encryption is explained in section 7.1.

4. Digital Signature

Using Digital signature a message can be signed by a device using its private key to ensure authenticity of the message. Any device that has got the access to the public key of the signed device can verify the signature. Thus the device receiving the message can ensure that the message is indeed signed by the intended device and is not modified during the transit. If any the data or signature is modified, the signature verification fails.

For e.g. if a device A need to ensure the authenticity of its message, the device A signs its message using its private key P_A. The device A will then send the message 'Msg' and signature 'Sgn' to device B. The device B, on receiving the message, can verify the message using A's public key U_A and there by ensuring that the message is indeed sent by A and is also not tampered during the transit. Since only the device A knows its private P_A key, it is impossible for any other device to forge the signature.

The examples of Digital Signature algorithms are RSA, DSA and ECDSA that are explained in sections 7.3, 8 and 11 respectively.

4.1. Certificate

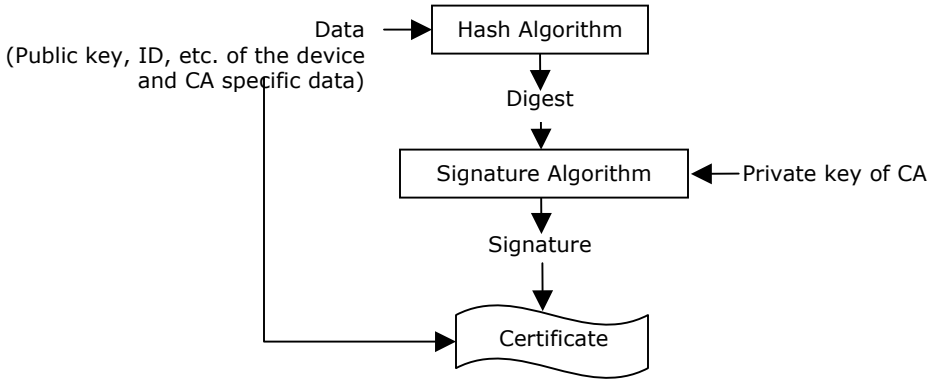
As seen in section 2, shared secret can be established between two devices using a key agreement algorithm by exchanging their public keys. However the credibility of received public key has to be ensured for a secured communication. For example consider two devices A and B establishing a shared secret. Both devices exchange their public keys. The devices calculate the shared secret using their private key and the other device's public key. Now consider an intermediate point H through which all the communication happens. If H captures B's public key and sends H's public key instead with B's identity, then A will end up in establishing shared secret with H and will communicate with H thinking that it is communicating with B. This happened because there is no way for A to verify that the received public key is indeed that of B. Here is where the Digital Certificate comes to play.

For data transfer in a network consider an authority trusted by all devices. This Trusted Certificate Authority (CA) signs the public keys and the unique identifiers of all devices. These signed data (public key, IDs etc.) along with the signature arranged in a standard format is called as the certificate. All the devices that take part in secured and trusted communication have to obtain a certificate from the trusted authority

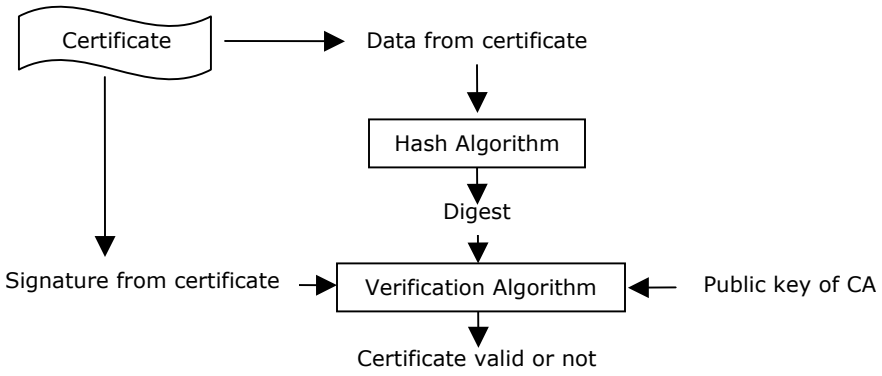
Now the device A and B exchanges their respective certificate instead of public key. These certificates are verified using CA's public key. Even if the intermediate point H modifies the public key or any other data in any of the certificate, the certificate verification will fail. The public keys of the CA are generally obtained as self-signed certificate.

Still the problem is not over. How to get the public key of the CA in a credible way? Since the CAs are few in numbers, the public key of the CA is obtained by some other trusted method. For example, in cases of secure Internet surfing the certificate of CA installed in the device along with the web browser.

The device that requires a certificate will send the certificate request to the CA. The request contains the device data such as device ID and device public key. The CA first finds the digest of the device data and CA specific data using a hash algorithm. CA then signs the hash using its private key and combines the data and signature in a standard format to form a certificate and is given to the device. The CA usually does some background check to ensure the device is not hostile before issuing the certificate. An example of a standard digital certificate format is X.509 certificates. [7]

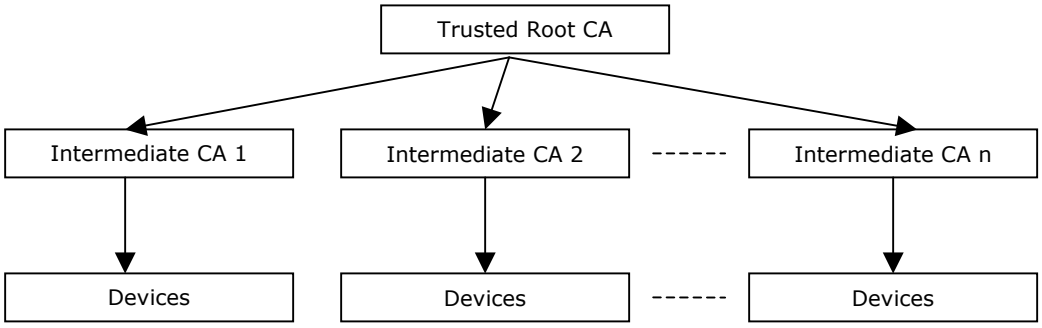


On receiving a certificate, a device extracts the data from the certificate, checks the ID and other data in the certificate. The signature in the certificate is verified using CA's public key.



4.2. Certificate Hierarchy

Trusting a certificate authority is a good idea for secured communication. But as the number of devices taking part in the communication increases and the location of these devices is distributed over different parts of the world, a central certificate authority may not suffice to issue and maintain the certificate of all the devices. Certificate hierarchy is the solution here.



In certificate Hierarchy there will be a trusted root CA who will give permission to other CAs to give certificate to the communicating devices. The root CA will issue the certificate to these intermediate CAs. These intermediate CAs then issue certificates to the device. In addition to issuing certificate to the devices the intermediate CA's will also give their respective certificate, issued by root CA, to the devices.

There can be multiple levels of certificate hierarchy in which the intermediate CAs will give permission to other CA to issue certificate to the communicating devices.

4.3. Verifying device certificate in Certificate Hierarchy

If a device A obtained a certificate from an intermediate CA, then it not only has its device certificate but also the certificate of the intermediate CA, which issued the certificate to the device A. If there are multiple levels of intermediate CA above the CA that issued certificate to the device, the device will have certificates of all intermediate CAs up to the root.

Consider another device B taking part in communication with device A. The device B on receiving the A's certificate may request A the certificate of intermediate CA who issued certificate to A. The device B may end up in asking all the certificate of intermediate CAs till the root CA for successful verification of A's device certificate. The device B must atleast have the self-signed root CA certificate obtained by a trusted means to successfully authenticate A's certificate.

4.4. Safe Storage of public keys

The public keys are generally stored as certificates and the root CA public key is stored as self signed certificate. Even though the security of public key cryptography does not depend on the secrecy of the public key, it is important that the public keys/certificates such as root CA certificate should be stored in a tamper resistant way. Public keys should be stored securely such that any third party cannot modify it. If Root CA certificate is modified then the attacker can make any certificate acceptable by the device thus defeating the certificate and secured communication.

4.5. One side Authentication

It may not be needed by both the devices participating in the secured communication to authenticate through Digital Certificate. In some client server architecture only the server needs to provide its certificate to prove its authenticity and hence to create secured channel. Once the secured channel is created the client authenticates/authorize itself by providing a 'user name and password'. In this case the server keeps the database of all the authorized users. Only if the username and password are matched the server gives out the 'secured' information.

One side authentication is typically done in web server. If user tries to access any secured website, the web site sends its certificate to the user. This is needed since the user does not want to send his user name and password to a non-trusted website. If the certificate cannot be verified the web browser will inform the user about the danger. On successful verification of the server certificate a secured channel is established and the user will give its username and password to the web server and this completes the authentication/authorization process.

4.6. Two side Authentication

There are communications in which the device doesn't keep any information such as username/password of the communicating device. This happens typically in peer-peer communication. In such cases both the devices need to have a Digital Certificate from a trusted authority. Both devices exchange their respective certificates for authentication during handshake of key agreement protocol.

5. Algorithms and Explanations

This section discusses a few public key algorithms and will also gives an explanation on how these algorithms work. The algorithms covered in this section are

- Key Agreement Algorithms – RSA, DH, ECDH
- Encryption Algorithms – RSA
- Signature Algorithms – RSA, DSA, ECDSA

This section also gives brief introduction to modular arithmetic.

5.1. Modular Arithmetic

Modular arithmetic is the commonly used arithmetic in public key cryptography. Modular arithmetic deals only with integers. Since it involves no floating-point operations, the mathematical calculations are more accurate and efficient than the real number arithmetic. Modular arithmetic over a number n involves arithmetic operations on integers between 0 and n – 1, where n is called the modulus. If the number happens to be out of this range in any of the operation the result, r, is wrapped around in to the range 0 and n – 1 by repeated subtraction of the modulus n from the result r. This is equivalent in taking the remainder of division operation r/n.

For e.g. for modulo 23 arithmetic

n=23, Let a=15, b=20

(a+b) mod n = (15+20) mod 23 = 35 mod 23 = 12

Since the result of a+b=35 which is out of the range [0,22], the result is wrapped around in to the range [0 22] by subtracting 35 with 23 till the result is in range [0,22].
a mod b is thus explained as remainder of division a/b.
Subtraction and multiplication can also be explained similarly.
A negative number is added repeatedly with n till it can be represented in the range [0,n-1]
The modular division a/b mod p is defined as $a*b^{-1}$ mod p. b^{-1} is the multiplicative inverse of b.
Multiplicative inverse of number b with respect to mod p is defined as a number b^{-1} such that $b*b^{-1}$ mod p = 1.

5.2. Congruent relation

Modular arithmetic is a congruent relation. Congruence is shown by the symbol '≡'. For a modulus n two numbers a and b are said to be congruent if a mod n = b mod n. i.e.
a ≡ b (mod n) if, a mod n = b mod n
For example consider the modulus 7 i.e. n = 7
Then the numbers 2, 9, 16, 23 etc are congruent to each other since
(2 mod 7) = (9 mod 7) = (16 mod 7) = (23 mod 7) etc

5.3. Properties of modular arithmetic

P1. a≡b mod n implies a–b=k*n, where k is an integer
P2. a mod n + b mod n ≡ a+b (mod n), also true for other operators '-', '*'and '/'
P3. a+b≡b+a (mod n) , also true for other operators '-', '*'and '/'
P4. a≡a mod n
P5. a≡b mod n implies b≡a mod n
P6. Fermat's little theorem, if M and p are coprime then $M^{p-1}\equiv 1 \pmod p$
P7. if p and q are coprime and also if a≡b(mod p) and a≡b(mod q) then a≡b(mod pq)

6. DH - Diffie-Hellman Key Agreement

Diffie-Hellman is a key agreement algorithm that helps two devices to agree on a shared secret between them with out the need to exchange any secret/private information. The DH standard is specified RFC 2631[4] . An overview of the algorithm is given below.

6.1. Key agreement Algorithm

For establishing shared secret between two device A and B, both device agrees on public constants p and g. Where p is a prime number and g is the generator less than p.

D1. Let a and b be the private keys of the devices A and B respectively, Private keys are random number less than p.
D2. Let g^amod p and g^bmod p be the public keys of devices A and B respectively
D3. A and B exchanged their public keys.
D4. The end A computes $(g^b \bmod p)^a$ mod p that is equal to g^{ba}mod p.
D5. The end B computes $(g^a \bmod p)^b$ mod p that is equal to g^{ab}mod p.
D6. Since K = g^{ba}mod p=g^{ab}mod p, shared secret = K.

6.2. DH - Mathematical Explanation

From the properties of modular arithmetic P2
a mod n * b mod n ≡ a * b (mod n)
Which can be written as
$(a_1 \bmod n)*(a_2 \bmod n)*\ldots*(a_k \bmod n) \equiv a_1 * a_2 * \ldots * a_k \pmod n$,
if a_i=a, where i = 1, 2, 3… k
$(a \bmod n)^k \equiv a^k \bmod n$,therefore ---- [HX1]
$(g^a \bmod p)^b \bmod p = g^{ab} \bmod p$ and
$(g^b \bmod p)^a \bmod p = g^{ba} \bmod p$
For all integers $g^{ab}=g^{ba}$,

Therefore shared secret $K=g^{ab} \bmod p = g^{ba} \bmod p$
Since it is practically impossible to find the private key a or b from the public key $g^a \bmod p$ or $g^b \bmod p$, it is not possible to obtain the shared secret K for a third party.

6.3. One-Way function in DH

For device A, Let a be the private key and $x = g^a \bmod p$ is the public key,

Here $x = g^a \bmod p$ is one-way function. The public key x is obtained easily in the forward operation, but finding 'a' given x, g and p is the reverse operation and takes exponentially longer time and is practically impossible. This is known as discrete logarithm problem[11].

7. RSA

RSA is a public key algorithm that is used for Encryption, Signature and Key Agreement. RSA typically uses keys of size 1024 to 2048. The RSA standard is specified RFC 3447, RSA Cryptography Specifications Version 2.1 [3]. Overviews of RSA algorithms are given below.

7.1. RSA Encryption

Parameter generation

R1. Select two prime numbers p and q.
R2. Find n=p*q, Where n is the modulus that is made public. The length of n is considered as the RSA key length.
R3. Choose a random number 'e' as a public key in the range $0<e<(p-1)(q-1)$ such that $\gcd(e,(p-1)(q-1))=1$.
R4. Find private key d such that $ed \equiv 1 \pmod{(p-1)(q-1)}$.

Encryption

Consider the device A that needs to send a message to B securely.

R5. Let e be B's public key. Since e is public, A has access to e.
R6. To encrypt the message M, represent the message as an integer in the range $0<M<n$.
R7. Cipher text $C = M^e \bmod n$, where n is the modulus.

Decryption

R8. Let C be the cipher text received from A.
R9. Calculate Message $M = C^d \bmod n$, where d is B's private key and n is the modulus.

7.2. RSA Key Agreement

Since public key cryptography involves mathematical operation on large numbers, these algorithms are considerably slow compared to the symmetric key algorithm. They are so slow that it is infeasible to encrypt large amount of data. Public key encryption algorithm such as RSA can be used to encrypt small data such as 'keys' used in private key algorithm. RSA is thus used as key agreement algorithm.

Key agreement algorithm

For establishing shared secret between two device A and B

R10. Generate a random number, key, at device A.
R11. Encrypt key by RSA encryption algorithm using B's public key and pass the cipher text to B
R12. At B decrypt the cipher text using B's private key to obtain the key.

7.3. RSA Signature

RSA Signature is similar to RSA encryption except that the private key is used for signing and public key is used for verification.

Parameter generation

The parameter generation process is same as that in RSA Encryption. See section 7.1

Signing

Consider the device A that needs to sign the data that it sends to B.

R13. Let d be A's private key
R14. To sign a data M, represent the data as an integer in the range $0<M<n$
R15. Signature $C = M^d \bmod n$

Verification

R16. Let M be the message and C be the signature received from A
R17. Calculate $M'=C^e \bmod n$, where e is A's public key. Since e is public, B has access to e
R18. If M'=M, the signature is verified, else failed.

7.4. One-Way function in RSA

Consider the key generation equation R4, $ed \equiv 1 (\mathrm{mod}(p-1)(q-1))$ and $n=p*q$
Where e is the public key d is the private key. p and q are kept private but n is made public. Since e is public, anybody who has access to p and q could easily generate the private key d using the above equation R4. The security of RSA depends on the difficulty to factorize n to obtain the prime numbers p and q. n is easily obtained by multiplying p and q but the reverse operation of factorizing n to obtain prime numbers p and q is practically impossible if p and q are sufficiently large numbers.

7.5. RSA – Mathematical Explanation

From parameter generation equation R4
$ed \equiv 1 \pmod{(p-1)(q-1)}$.
From the encryption equation R7
Cipher text $C=M^e \bmod n$
From the decryption equation R9
Message $M=C^d \bmod n$
Combining above two equations $M = (M^e \bmod n)^d \bmod n$, using equation HX1
$M=M^{ed} \bmod n$
Similarly by combining signature and verification equation R15 and R17 we get
$M=M^{ed} \bmod n$

So to prove the correctness of RSA, it has to prove that
$M=M^{ed} \bmod n$, if
$ed \equiv 1 \pmod{(p-1)(q-1)}$

From the above equation $ed \equiv 1 \pmod{(p-1)(q-1)}$ and property P1 it follows that
$ed-1=K(p-1)(q-1)$, which can also be written as
$ed-1=k(p-1)$, and ---- [RX1]
$ed-1=k'(q-1)$ ---- [RX2]
Where K, k and k' are positive integers

Since any integer is congruent to itself it can be written as
$M^{ed} \equiv M^{ed} \pmod p$. i.e.
$M^{ed} \equiv M^{ed-1} * M \pmod p$,
Using equation RX1 the above equation can be written as

$M^{ed} \equiv M^{k(p-1)} * M \pmod{p}$, ---- [RX3]
Since p is prime, any integer M can either be a co-prime with p or a multiple of p.

Case 1: If M and p are coprime, then from Fermat's little theorem
$M^{p-1} \equiv 1 \pmod{p}$, or
$M^{k(p-1)} \equiv 1^k \pmod{p}$, i.e.
$M^{k(p-1)} \equiv 1 \pmod{p}$ ---- [RX4]
From equations RX3 and RX4
$M^{ed} \equiv M \pmod{p}$

Case 2: If M is a multiple of p, then M^{ed} will also be a multiple of p, i.e.
M mod p=0, also M^{ed} mod p = 0, thus from congruence relation,
$M^{ed} \equiv M \pmod{p}$
Similarly using RX2 it can be proved that $M^{ed} \equiv M \pmod{p}$ for above two cases.

Since p and q are prime numbers they are coprime to each other. Therefore by using property P7 the above two equations can be combined as
$M^{ed} \equiv M \pmod{p*q}$, by property P5
$M \equiv M^{ed} \pmod{p*q}$
Since M is chosen in the range 0 and (p*q–1)
$M = M^{ed} \bmod p*q$, i.e. $M = M^{ed} \bmod n$

8. DSA – Digital Signature Algorithm

DSA is a public key algorithm that is used for Digital Signature. The DSA standard is specified FIPS 186-2, Digital Signature Standard [2] An overview of the algorithm is given below.

Parameter generation

S1. Choose a 160-bit prime q.
S2. For an integer z, choose an L-bit prime p, such that $p=qz+1$, $512 \le L \le 1024$, and L is divisible by 64.
S3. Choose h, where $1<h<p-1$ such that $g=h^Z \bmod p>1$.
S4. Choose a random number x, where $0<x<q$.
S5. Calculate $y=g^x \bmod p$.
S6. Public key is (p, q, g, y). Private key is x.

Signing

Consider the device A that sign the data M that it sends to B.

S7. Let x be A's private key and (p, q, g, y) be A's public key.
S8. Generate a random per-message value k, where $0<k<q$.
S9. Calculate $r = (g^k \bmod p) \bmod q$.
S10. Calculate $s = (k^{-1}(M+x*r)) \bmod q$, where M is the hash SHA1 of the message
S11. The signature is (r, s).

Verification

S12. Let M be the message and (r, s) be the signature received from A
S13. Let (p, q, g, y) be A's public key. Since (p, q, g, y) is public, B has access to it.
S14. Calculate $w = s^{-1} \bmod q$.
S15. Calculate u1 = (M*w) mod q, where M is the hash SHA1 of the message.
S16. Calculate u2 = (r*w) mod q.
S17. Calculate $v = ((g^{u1} * y^{u2}) \bmod p) \bmod q$.
S18. The signature is valid if v=r, invalid otherwise.

8.1. DSA - Mathematical Explanation

From S18, Signature is valid if v=r, to prove the correctness of the algorithm it has to prove that v = r if signature is valid.
Form S17, $v = ((g^{u1}*y^{u2}) \bmod p) \bmod q$.
But from S5, $y=g^x \bmod p$, i.e $y \equiv g^x \bmod p$, i.e. using equation HX1, $y^{u2} \equiv g^{x*u2} \bmod p$
Therefore
$v = ((g^{u1}*g^{x.u2}) \bmod p) \bmod q$. ---- [DX1]
But from S16, $g^{x.u2}=g^{x*(r*w \bmod q)}=g^{(x \bmod q)*(r*w \bmod q)}$
From P2 $(x \bmod q)*(r*w \bmod q) \equiv (x*r*w) \bmod q$
Using P1 $(x \bmod q)*(r*w \bmod q)= (x*r*w) \bmod q + k*q$
Where k is an integer, Therefore
$g^{x.u2}=g^{(x*r*w) \bmod q + k*q}$, i.e.
$g^{x.u2}=(g^{(x*r*w) \bmod q}) *(g^{k*q})$ ---- [DX2]
From S3, $g = h^z \bmod p$,
Using HX1 and S2, $g^q \equiv h^{qz} \equiv h^{p-1} \pmod p$
By P6, Fermat's little theorem $h^{p-1} \equiv 1 \pmod p$, i.e. $g^q \equiv 1 \pmod p$
Substituting the value of $g^q \equiv 1 \pmod p$ in DX2
$g^{x.u2}=(g^{(x*r*w) \bmod q})$ ---- [DX3]
But from S15, $g^{u1}= g^{(M*w \bmod q)}$ ---- [DX4]
Substituting DX3 and DX4 in DX1
$v=((g^{wM \bmod q + xrw \bmod q}) \bmod p) \bmod q$. i.e.
$v=((g^{w\,(M+xr) \bmod q}) \bmod p) \bmod q$. ------ [DX5]
From S10, in signature algorithm
$s=(k^{-1}(M+x*r)) \bmod q$ i.e.
$k \equiv s^{-1}\,(M+x*w) \bmod q$ ----- [DX6]
But from S14, $w=s^{-1} \bmod q$ i.e.
$s^{-1} \equiv w \bmod q$
Therefore equation DX6 can be written as
$k \equiv w(M+x*w) \bmod q$
Since $k<q$, $k=w(M+x*w) \bmod q$ ----- [DX7]
Combining DX5 and DX7 and using S9
$V=((g^k) \bmod p) \bmod q=r$. i.e. $v=r$

8.2. One-Way function in DSA

Consider the equation S5, $\mathbf{y=g^x \bmod p}$, where x is the private key but y, g and p are public. Calculating y from g, x and p is a forward operation but obtaining x from the given y g and p is the reverse operation and hence finding x is impossible for large numbers. This is known as discrete logarithm problem[11] .

9. Elliptic Curve Cryptography

Elliptic curve cryptography (ECC) is relatively new technology compared to other public key cryptography such as RSA. Elliptic key operates on smaller key size. A 160-bit key in ECC is considered to be as secured as a 1024 bit key in RSA. ECC operates on the points in the elliptic curve $\mathbf{y^2=x^3+ax+b}$, where $\mathbf{4a^3+27b^2 \neq 0}$.

The above equation of elliptic curve is in real coordinate. To make elliptic curve operation efficient and accurate the elliptic curve can be defined in finite fields. Elliptic curve in two finite fields, prime field and binary field, are defined by standard. In prime field operation the elliptic curve equation is modified as $\mathbf{y^2 \bmod p=x^3+ax+b \bmod p}$, where $\mathbf{4a^3+27b^2 \bmod p \neq 0}$. The ECC standards are specified in SEC, Standards for Efficient Cryptography [5]

9.1. Domain parameters

There are certain public constants that are shared between parties involved in secured and trusted ECC communication. This includes curve parameter **a**, **b**, a generator point **G** in the

chosen curve, the modulus **p**, order of the curve **n** and the cofactor **h**. There are several standard domain parameters defined by SEC, Standards for Efficient Cryptography[6] .

9.2. Point multiplication

Point multiplication is the central operation in ECC. In point multiplication a point P on the elliptic curve is multiplied with a scalar k using elliptic curve equation to obtain another point Q on the same elliptic curve.
i.e. k*P=Q
Point multiplication is achieved by two basic elliptic curve operations

- **Point addition**, adding two points J and K using elliptic curve equation to obtain another point L i.e., L = J + K.
- **Point doubling**, adding a point J to itself using elliptic curve equation to obtain another point L i.e. L = 2J.

Here is a simple example of point multiplication.
Let P be a point on an elliptic curve. Let k be a scalar that is multiplied with the point P to obtain another point Q on the curve. i.e. to find Q = k*P.
If k = 23 then k*P = 23*P = 2(2(2(2P) + P) + P) + P.
In the ECC explanations given below upper case letter indicates a point in the elliptic curve and the lower case letter indicates a scalar

9.3. One Way function in ECC

The security of ECC depends on the difficulty of Elliptic Curve Discrete Logarithm Problem. Let P and Q be two points on an elliptic curve such that k*P = Q, where k is a scalar. Q can be easily obtained from P and k but given P and Q, it is computationally infeasible to obtain k, if k is sufficiently large. k is the discrete logarithm of Q to the base P.

10. ECDH – Elliptic curve Diffie Hellman

ECDH, a variant of DH, is a key agreement algorithm.
For generating a shared secret between A and B using ECDH, both have to agree up on Elliptic Curve domain parameters. An overview of ECDH is given below.

Key Agreement Algorithm

For establishing shared secret between two device A and B

E1. Let d_A and d_B be the private key of device A and B respectively, Private keys are random number less than n, where n is a domain parameter.
E2. Let $Q_A = d_A*G$ and $Q_B = d_B*G$ be the public key of device A and B respectively, G is a domain parameter
E3. A and B exchanged their public keys
E4. The end A computes $K = (x_K, y_K) = d_A*Q_B$
E5. The end B computes $L = (x_L, y_L) = d_B*Q_A$
E6. Since K=L, shared secret is chosen as x_K

10.1. ECDH - Mathematical Explanation

To prove the agreed shared secret K and L at both devices A and B are the same
From E2, E4 and E5
$K = d_A*Q_B = d_A*(d_B*G) = (d_B*d_A)*G = d_B*(d_A*G) = d_B*Q_A = L$
Hence K = L, therefore $x_K = x_L$
Since it is practically impossible to find the private key d_A or d_B from the public key Q_A or Q_B, its not possible to obtain the shared secret for a third party.

11. ECDSA - Elliptic curve Digital Signature Algorithm

ECDSA is a variant of the Digital Signature Algorithm (DSA). For sending a signed message from A to B, both have to agree up on Elliptic Curve domain parameters. Sender A have a

key pair consisting of a private key d_A (a randomly selected integer less than n, where n is the order of the curve, an elliptic curve domain parameter) and a public key $Q_A = d_A*G$ (G is the generator point, an elliptic curve domain parameter). An overview of ECDSA process is defined below.

Signing

Consider the device A that signs the data M that it sends to B.

E7. Let d_A be A's private key
E8. Calculate m = HASH (M), where HASH is a hash function, such as SHA-1
E9. Select a random integer k such that $0<k<n$
E10. Calculate $r = x_1 \bmod n$, where $(x_1, y_1) = k*G$
E11. Calculate $s = k^{-1}(m + d_A*r) \bmod n$
E12. The signature is the pair (r, s)

Verification

E13. Let M be the message and (r, s) be the signature received from A
E14. Let Q_A be A's public key. Since Q_A is public, B has access to it.
E15. Calculate m = HASH (M)
E16. Calculate $w = s^{-1} \bmod n$
E17. Calculate $u_1 = m*w \bmod n$ and $u_2 = r*w \bmod n$
E18. Calculate $(x_1, y_1) = u_1*G + u_2*Q_A$
E19. The signature is valid if $x_1 = r \bmod n$, invalid otherwise

11.1. ECDSA - Mathematical Explanation

From the verification equation E19, the signature is valid if $x_1=r \bmod n$ ---- [EX1]
But from E18, x_1 is the x-coordinate of equation $u_1*G+u_2*Q_A$
From E10 r is the x-coordinate of equation k*G
Thus to prove equation EX1, It has to prove that
$u_1*G+u_2*Q_A = k*G$
Substituting the value of u_1 and u_2 from E17 the first part of the above equation
$u_1*G+u_2*Q_A = (m*w \bmod n)*G + (r*w \bmod n)*Q_A$
But $Q_A=d_A*G$, therefore
$u_1*G+u_2*Q_A = (m*w \bmod n)*G + (r*w*d_A \bmod n)*G$, i.e.
$u_1*G+u_2*Q_A = (w*(m+r*d_A) \bmod n)*G$
But from E16, $w=s^{-1} \bmod n$, i.e $s^{-1} \equiv w \bmod n$ therefore
$u_1*G+u_2*Q_A = (s^{-1}*(m+r*d_A) \bmod n)*G$ ----- [EX2]
but from equation E11
$s=k^{-1}(m+d_Ar) \bmod n$, i.e. $k \equiv s^{-1}(m+d_Ar) \bmod n$
Substituting k in EX2
$u_1*G + u_2*Q_A = k*G$
Therefore $x_1=r \bmod n$

12. Conclusion

Public key cryptography is an innovation and is an unavoidable part of almost all security protocol and application. Being able to negotiate a shared secret between two devices online with out the need of any exchange of secret data created a breakthrough in secure network/internet communication. Though theoretically it is possible to find the shared secret from the available public information, it will take exponentially longer time making it practically impossible. It is the belief in age-old mathematics, that finding an easy method for reverse process of one-way function is unlikely, keeps the public key cryptography going.

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